

## Problem 2.12

[Difficulty: 3]

**2.12** The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{Ky}{2\pi(x^2 + y^2)}\hat{i} + \frac{Kx}{2\pi(x^2 + y^2)}\hat{j}$$

where  $K = 10^5 \text{ m}^2/\text{s}$ , and the  $x$  and  $y$  coordinates are parallel to the local latitude and longitude. Plot the velocity magnitude along the  $x$  axis, along the  $y$  axis, and along the line  $y = x$ , and discuss the velocity direction with respect to these three axes. For each plot use a range  $x$  or  $y = -1 \text{ km}$  to  $1 \text{ km}$ , excluding  $|x|$  or  $|y| < 100 \text{ m}$ . Find the equation for the streamlines and sketch several of them. What does this flow field model?

**Given:** Flow field

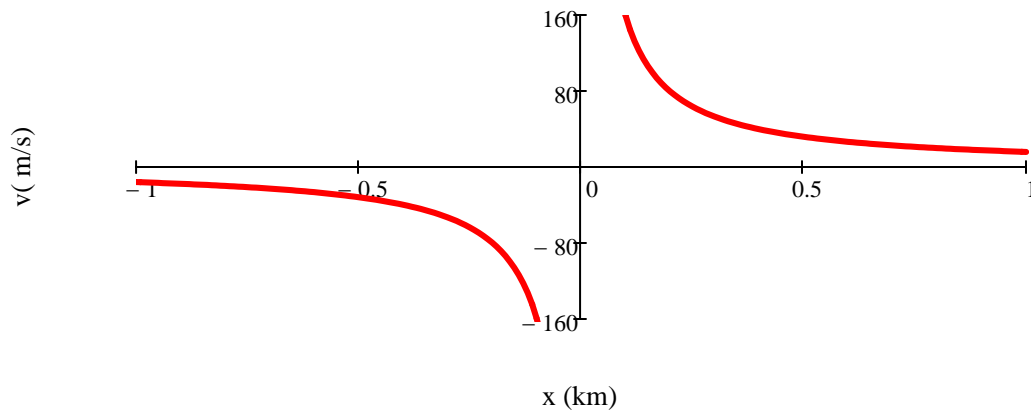
**Find:** Plot of velocity magnitude along axes, and  $y = x$ ; Equation of streamlines

**Solution:**

On the  $x$  axis,  $y = 0$ , so

$$u = -\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)} = 0 \quad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)} = \frac{K}{2 \cdot \pi \cdot x}$$

Plotting



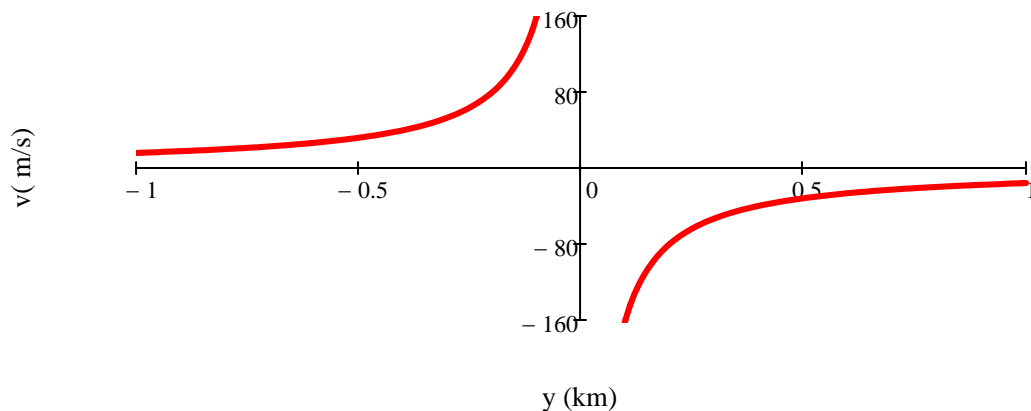
The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.

This can also be plotted in Excel.

On the  $y$  axis,  $x = 0$ , so

$$u = -\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)} = -\frac{K}{2 \cdot \pi \cdot y} \quad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)} = 0$$

Plotting



The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero. This can also be plotted in Excel.

On the  $y = x$  axis

$$u = -\frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + x^2)} = -\frac{K}{4 \cdot \pi \cdot x} \quad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + x^2)} = \frac{K}{4 \cdot \pi \cdot x}$$

The flow is perpendicular to line  $y = x$ :

Slope of line  $y = x$ : 1

Slope of trajectory of motion:  $\frac{u}{v} = -1$

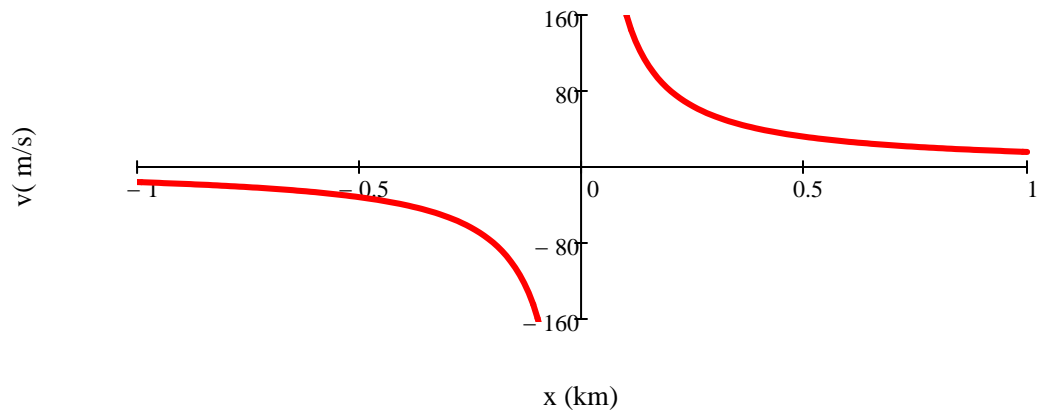
If we define the radial position:

$$r = \sqrt{x^2 + y^2} \quad \text{then along } y = x \quad r = \sqrt{x^2 + x^2} = \sqrt{2} \cdot x$$

Then the magnitude of the velocity along  $y = x$  is

$$V = \sqrt{u^2 + v^2} = \frac{K}{4 \cdot \pi} \cdot \sqrt{\frac{1}{x^2} + \frac{1}{x^2}} = \frac{K}{2 \cdot \pi \cdot \sqrt{2} \cdot x} = \frac{K}{2 \cdot \pi \cdot r}$$

Plotting



This can also be plotted in Excel.

For streamlines

$$\frac{v}{u} = \frac{dy}{dx} = \frac{\frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)}}{-\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)}} = -\frac{x}{y}$$

So, separating variables

$$y \cdot dy = -x \cdot dx$$

Integrating

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

The solution is

$$x^2 + y^2 = C \quad \text{which is the equation of a circle.}$$

Streamlines form a set of concentric circles.

This flow models a vortex flow. See Example 5.6 for streamline plots. Streamlines are circular, and the velocity approaches infinity as we approach the center. In Problem 2.11, we see that the streamlines are also circular. In a real tornado, at large distances from the center, the velocities behave as in this problem; close to the center, they behave as in Problem 2.11.